*Journal of Statistical Physics, Vol. 109, Nos. 1/2, October 2002 (© 2002)*

# Knotted Paths in Percolation

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*Received February 1, 2002; accepted April 12, 2002*

We study the topology of doubly-infinite paths in the bond percolation model on the three-dimensional cubic lattice. We propose a natural definition of a knotted doubly-infinite path. We prove the existence of a critical probability $p_k$ satisfying $p_c < p_k < 1$ (where $p_c$ is the usual percolation critical probability), such that for $p_c < p < p_k$, all doubly-infinite open paths are knotted, while for $p > p_k$ there are unknotted doubly-infinite paths.

**KEY WORDS:** Percolation; random knot; enhancement; critical probability.

## 1. INTRODUCTION

Knotting of random paths has important applications in polymer science, and has been extensively studied. Previous work has involved knotting probabilities of finite self-avoiding walks and polygons, chosen according to various random mechanisms. For details, and for information on the physical applications, the reader is referred to the articles in ref. 1, for example. For more information on knot theory see ref. 2.

Here we consider a closely related problem. In the percolation model, edges of the infinite three-dimensional cubic lattice are declared *open* with probability $p$, or *closed* with probability $1-p$, independently for different edges. (For more details of percolation see ref. 3). We consider the question: when do there exist open knotted or unknotted doubly-infinite paths?

As in the case of entanglement (see refs. 4–7) it is not immediately obvious how to give a "correct" definition of a knotted doubly-infinite path. The situation is complicated by the possibility that a path may "double back" to untie a potential knot, as in Fig. 1. We shall not pursue

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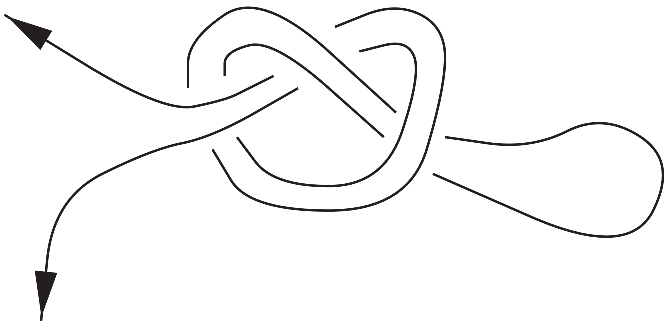

Fig. 1. This path is unknotted.

the question of possible definitions in detail; instead we shall give one natural definition, noting that there may be others.

Standard results imply that when $p$ is greater than the percolation critical probability $p_c$, there exist open doubly-infinite paths. We shall prove that for $p$ sufficiently close to $p_c$ all such paths are knotted, while for $p$ sufficiently close to 1 there are unknotted doubly-infinite paths.

## 2. NOTATION AND RESULTS

We start with some definitions. The three-dimensional cubic lattice is the graph with vertex set $\mathbb{Z}^3$ and edge set

$$\mathbb{L} = \{\{x, y\} \subseteq \mathbb{Z}^3 : \|x - y\| = 1\}$$

where $\|\cdot - \cdot\|$ denotes Euclidean distance. The *origin* is the vertex $O = (0, 0, 0) \in \mathbb{Z}^3$. In the bond percolation model with parameter $p$, each edge in $\mathbb{L}$ is declared *open* with probability $p$, and *closed* otherwise, independently for different edges. More formally, we consider the product probability measure $P_p$ on the probability space $\{0, 1\}^{\mathbb{L}}$. An element $\omega$ of the probability space is called a *configuration*, and an edge $e \in \mathbb{L}$ is said to be *open* if $\omega(e) = 1$ and *closed* if $\omega(e) = 0$. We write $W = W(\omega)$ for the random set of all open edges.

A *finite path* is a non-empty set of edges of the form $\{\{x_0, x_1\}, \{x_1, x_2\},..., \{x_{r-1}, x_r\}\}$, and a *doubly-infinite path* is a set of edges of the form $\{..., \{x_{-1}, x_0\}, \{x_0, x_1\}, \{x_1, x_2\},...\}$, where in both cases the $x_i$ are pairwise distinct vertices. A *subpath* is a subset of a path which is itself a path.

Percolation theory is concerned with the existence of infinite connected components. We define

$$\theta(p) = P_p(W \text{ has an infinite connected component containing } O)$$

and

$$p_c = \sup\{p: \theta(p) = 0\}.$$

It is known (by the results in ref. 8, for example) that for all $p > p_c$ there exist doubly-infinite open paths almost surely. For more information on percolation see ref. 3.

Our aim here is to study knotting of paths, and for this we require the following topological definitions. A *ball* $B$ is a subset of $\mathbb{R}^3$ which is homeomorphic to $\{x \in \mathbb{R}^3 : \|x\| \leqslant 1\}$, and the *boundary* $\partial B$ of a ball is the image of $\{x \in \mathbb{R}^3 : \|x\| = 1\}$ under such a homeomorphism. Similarly an arc $\alpha$ is a subset of $\mathbb{R}^3$ homeomorphic to $[-1, 1] \times \{0\}^2$, and $\partial\alpha$ is the image of $\{-1, 1\} \times \{0\}^2$. The following definitions relating to ball-arc pairs are standard; for more details see ref. 2. A *ball-arc pair* is a pair $(B, \alpha)$, where $B$ is a ball and $\alpha$ is an arc, such that $\alpha \subseteq B$ and $\alpha \cap \partial B = \partial\alpha$. Two ball-arc pairs $(B, \alpha)$ and $(B', \alpha')$ are *equivalent* if there is a homeomorphism from $B$ to $B'$ which maps $\alpha$ to $\alpha'$. A ball-arc pair is said to be *unknotted* if it is equivalent to the ball-arc pair $([-1, 1]^3, [-1, 1] \times \{0\}^2)$, and *knotted* otherwise. (Note that *any* arc forms an unknotted ball-arc pair with some ball; see ref. 2 for more details).

For an edge $e = \{x, y\} \in \mathbb{L}$ we denote by $[e]$ the closed line segment

$$[e] = \{\lambda x + (1 - \lambda)\, y : \lambda \in [0, 1]\} \subseteq \mathbb{R}^3.$$

For a set of edges $G$ we write $[G] = \bigcup_{e \in G} [e] \subseteq \mathbb{R}^3$. By a *block* we mean a ball of the form $[a, b] \times [c, d] \times [e, f]$, where $a,..., f$ are integers. Let $F$ be a *finite* path. We say $F$ is *neat* if there exists a block $B$ such that $(B, [F])$ is an unknotted ball-arc pair. We say that a doubly-infinite path $G$ is *unknotted* if every finite subpath of $G$ is a subpath of some neat finite subpath of $G$, and *knotted* otherwise.

We now define

$$\kappa(p) = P_p(\text{there is an open unknotted doubly-infinite path containing } O).$$

It is easy to see that $\kappa$ is an increasing function, so we define

$$p_k = \sup\{p: \kappa(p) = 0\}.$$

**Theorem.** We have

$$p_c < p_k < 1.$$

It follows from the theorem that if $p_c < p < p_k$, then every doubly-infinite path is knotted almost surely.

## 3. PROOF OF THEOREM

We begin with the latter inequality of the theorem. We say that a doubly-infinite path $\{..., \{x_{-1}, x_0\}, \{x_0, x_1\},...\}$ is *oriented* if $(x_{i+1})_j \geqslant (x_i)_j$ for all $i$ and each $j = 1, 2, 3$, where $(x_i)_j$ denotes the $j$-coordinate of the 3-vector $x_i$. Standard results imply that for $p$ sufficiently close to unity, $O$ is contained in an open oriented doubly-infinite path with positive probability (see ref. 3, Section 12.8). The inequality $p_k < 1$ therefore follows from the observation (which we justify below) that every oriented doubly-infinite path is unknotted.

To justify the claim above, note that it is sufficient to prove that any finite subpath of an oriented doubly-infinite path is neat. Let $F = \{\{x_0, x_1\},..., \{x_{r-1}, x_r\}\}$ be such a path. Clearly we may find a block $B$ such that $(B, [F])$ is a ball-arc pair (we start with the block having opposite corners $x_0$ and $x_r$, and then enlarge it to ensure that $[F] \cap \partial B = \partial[F] = \{x_0, x_r\}$). For $x \in \mathbb{R}^3$ define $\phi(x) = x_1 + x_2 + x_3$. Note that $\phi(x_i)$ is strictly monotonic in $i$, increasing (or decreasing) by 1 as $i$ increased by 1. Let $L$ be the straight line segment joining $x_0$ and $x_r$. It is straightforward to show that $(B, [F])$ is equivalent to $(B, L)$; there is a suitable piecewise-linear homeomorphism which preserves $\phi(x)$ for all $x \in B$, and is the identity on $\partial B$. It is now easily seen (by applying a further homeomorphism) that $(B, L)$ is an unknotted ball-arc pair, and hence $(B, [F])$ is also.

We now turn to the former inequality of the theorem. Let $C = [0, 4] \times [0, 5] \times [0, 4]$, let $H$ be the set of all edges of $\mathbb{L}$ having both vertices in $C$, and let $K$ be the subset of $H$ illustrated in Fig. 2 (the outline of $C$ is also illustrated). Standard tools of knot theory may be used to show that $(C, [K])$ is a knotted ball-arc pair (for example, using the Jones polynomial, see ref. 2). We define a "diminishment" of $W$ as follows. Given $\omega$, define

$$W' = W \Big\backslash \bigcup_{\substack{x \in \mathbb{Z}^3: \\ W \cap (H+x) = K+x}} (K+x);$$

that is, $W'$ is obtained from $W$ by deleting translated copies of Fig. 2 wherever they occur.

The following is a consequence of a slight modification of results in ref. 9. There exists an interval $[p_1, p_2]$ where $p_1 < p_2$ such that for $p \in [p_1, p_2]$ we have

$$P_p(W \text{ has an infinite connected component}) = 1 \tag{1}$$

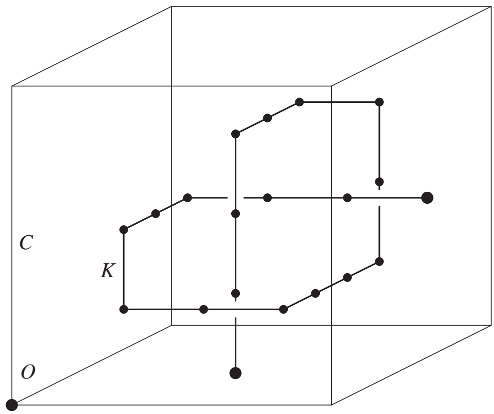


Fig. 2. The path $K$. The ends of the path lie on the boundary of the block $C$, while all the other vertices lie in its interior.

but

$$P_p(W' \text{ has an infinite connected component}) = 0. \tag{2}$$

The main result in ref. 9 is for "enhancements"—systematic alterations involving the addition of edges, whereas the construction of $W'$ is a "diminishment" involving removal of edges. The necessary modifications to the proof in ref. 9 are straightforward. A diminishment was also used in ref. 10; see also ref. 3, p. 65.

Now, (1) implies that $p_c \leqslant p_1$. And (2) implies that, $P_p$-a.s. for $p \in [p_1, p_2]$, $W'$ contains no unknotted doubly-infinite path. We shall show that this in turn implies that $W$ has no unknotted doubly-infinite path, and therefore $p_2 \leqslant p_k$, establishing the required inequality.

We must show that there exists no $\omega$ for which $W$ contains an unknotted doubly-infinite path but $W'$ does not. Suppose on the contrary that for some $\omega$, $U$ is an unknotted doubly-infinite path which is a subset of $W$ but not of $W'$. Clearly, $U$ must have a subpath of the form $K+x$, and without loss of generality we may assume that $x = O$, so that $K \subseteq U$ and $W \cap H = K$. Since $U$ is unknotted, $K$ must lie in a neat subpath of $U$, so consider a block $B$ and a finite path $L$ satisfying $K \subseteq L \subseteq U$ such that $(B, [L])$ is an unknotted ball-arc pair. We shall use standard tools from knot theory to show that this is impossible; detailed justification of some of the steps may be found in ref. 2. First add a "point at infinity" to $\mathbb{R}^3$ making it into a 3-sphere. For any ball $A$, we write $\hat{A}$ for the closure of its complement in $\mathbb{R}^3 \cup \{\infty\}$; this is also a ball. Now, since $K \subseteq L$, it may be seen by inspecting Fig. 2 that we must have $H \subseteq B$. We may find an arc $\beta \in \hat{B}$ such that $\partial\beta = \partial[L]$ and $(\hat{B}, \beta)$ is an unknotted ball-arc pair. It follows that $\beta \cup [L]$ is an unknotted loop (see ref. 2 for a definition). We can consider $\beta \cup [L]$ as the union of the arcs $[K]$ and $\beta \cup [L \setminus K]$; but $(H, [K])$ is a *knotted* ball-arc pair, and $(\hat{H}, \beta \cup [L \setminus K])$ is a ball-arc pair (because $L \cap H = K$). This contradicts a standard theorem which states that no knot has an additive inverse (Corollary 2.5 in ref. 2).

## ACKNOWLEDGMENTS

Research funded in part by NSF Grant DMS-0072398.

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